Logarithmic vector-valued modular forms and polynomial-growth estimates of their Fourier coefficients

Marvin Knopp
Temple University
Geoffrey Mason*
University of California at Santa Cruz

Abstract

We establish (Theorem 3.6) polynomial-growth estimates for the Fourier coefficients of holomorphic logarithmic vector-valued modular forms. (MSC2010: 11F12, 11F99)

1 Introduction

The present work is a natural sequel to our earlier articles on 'normal' and 'logarithmic' vector-valued modular forms [KM1], [KM2], [KM3]. The component functions of a normal vector-valued modular form F are ordinary left-finite q-series with real exponents. Equivalently, the finite-dimensional representation ρ associated with F has the property that $\rho(T)$ is (similar to) a matrix that is unitary and diagonal. Here, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

In the case of a general representation, $\rho(T)$ is not necessarily diagonal but may always be assumed to be in Jordan canonical form[1]. This circumstance leads to *logarithmic*, or *polynomial* q-expansions for the component functions of a vector-valued modular form associated to ρ (see Subsection 2.2), which take the form

$$f(\tau) = \sum_{j=0}^{t} (\log q)^j h_j(\tau), \tag{1}$$

where the $h_j(\tau)$ are ordinary q-series. There follow naturally the definition of logarithmic vector-valued modular form and the concomitant notions of logarithmic meromorphic, holomorphic (i.e., entire in the sense of Hecke) and cuspidal vector-valued modular forms (Subsection 2.3).

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[1]We actually use a modified Jordan canonical form. See [KM3] for details.

In [KM3] we derived a number of the properties of logarithmic vector-valued modular forms (LVVMF's) by introducing appropriate Poincaré series. In [KM2] we elaborated a well-known method of Hecke [H] devised to obtain polynomial-growth estimates of the Fourier coefficients of classical (i.e. scalar) holomorphic modular forms, and by this means we derived analogous estimates for the coefficients of *normal* VVMF's. The purpose of the present note is to extend Hecke's method even further to establish similar polynomial-growth estimates for the coefficients of holomorphic (i.e. entire in the sense of Hecke), including cuspidal, LVVMF's. Our extension of the method here entails the assumption that the eigenvalues of $\rho(T)$ have absolute value 1, so that the q-series $h_j(\tau)$ in (1) again have real exponents, a condition that will be assumed implicitly in the remainder of the article. It requires as well a simple new estimate (Proposition 3.3) that we apply in §3.2. (This same estimate is an important ingredient in our proof of convergence of the logarithmic Poincaré series introduced in [KM3].)

The occurrence of q-expansions of the form (1) is well known in rational and logarithmic conformal field theory. Indeed, much of the motivation for the present work originates from a need to develop a systematic theory of vector-valued modular forms wide enough in scope to cover possible applications in such field theories. By results in [DLM] and [M], the eigenvalues of $\rho(T)$ for the representations that arise in rational and logarithmic conformal field theory are indeed of absolute value 1 (in fact, they are roots of unity). Thus this assumption is natural from the perspective of conformal field theory. Our earlier results [KM1] on polynomial estimates for Fourier coefficients of entire vector-valued modular forms in the normal case have found a number of applications to the theory of rational vertex operator algebras, and we expect that the extension to the logarithmic case that we prove here will be useful in the study of C_2-cofinite vertex operator algebras, which constitute the algebraic underpinning of logarithmic field theory.

Other properties of logarithmic vector-valued modular forms are also of interest, from both a foundational and applied perspective. These include a Petersson pairing, generation of the space of cusp-forms by Poincaré series, existence of a natural boundary for the component functions, and explicit formulas (in terms of Bessel functions and Kloosterman sums) for the Fourier coefficients of Poincaré series. This program was carried through in the normal case in [KM2]. It is evident that the more general logarithmic case will yield a similarly rich harvest, but one must expect more complications. For example, there are logarithmic vector-valued modular forms with nonconstant component functions that may be extended to the whole of the complex plane, so that the usual natural boundary result is false *per se*. Such logarithmic vector-valued modular forms are studied (indeed, classified) in [KM4]. Furthermore, our preliminary calculations indicate that the explicit formulas exhibit genuinely new features. We hope to return to these questions in the future.

2 Logarithmic vector-valued modular forms

For the sake of completeness and clarity we present here much of the introductory material on LVVMF's that appears in [KM3].

2.1 Unrestricted vector-valued modular forms

We start with some notation that will be used throughout. The *modular group* is

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$

It is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

The complex upper half-plane is

$$\mathfrak{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}.$$

There is a standard left action $\Gamma \times \mathfrak{H} \to \mathfrak{H}$ given by Möbius transformations:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) \mapsto \frac{a\tau + b}{c\tau + d}.$$

Let $\mathfrak{F}$ be the space of holomorphic functions in $\mathfrak{H}$. There is a standard 1-cocycle $j : \Gamma \to \mathfrak{F}$ defined by

$$j(\gamma, \tau) = j(\gamma)(\tau) = c\tau + d, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$\rho : \Gamma \to GL(p, \mathbb{C})$ will always denote a p-dimensional matrix representation of Γ. An *unrestricted vector-valued modular form of weight k with respect to ρ* is a holomorphic function $F : \mathfrak{H} \to \mathbb{C}^p$ satisfying

$$\rho(\gamma)F(\tau) = F|_k\gamma(\tau), \quad \gamma \in \Gamma,$$

where the right-hand-side is the usual stroke operator

$$F|_k\gamma(\tau) = j(\gamma, \tau)^{-k}F(\gamma\tau). \tag{3}$$

We could take $F(\tau)$ to be *meromorphic* in $\mathfrak{H}$, but we will not consider that more general situation here. Choosing coordinates, we can rewrite (3) in the form

$$\rho(\gamma) \begin{pmatrix} f_1(\tau) \\ \vdots \\ f_p(\tau) \end{pmatrix} = \begin{pmatrix} f_1|_k\gamma(\tau) \\ \vdots \\ f_p|_k(\gamma)(\tau) \end{pmatrix} \tag{4}$$

with each $f_j(\tau) \in \mathfrak{F}$. We also refer to (F, ρ) as an unrestricted vector-valued modular form.

2.2 Logarithmic q-expansions

In this Subsection we consider the q-expansions associated to unrestricted vector-valued modular forms. We make use of the polynomials defined for $k \geq 1$ by

$$\binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!},$$

and with $\binom{x}{0} = 1$ and $\binom{x}{k} = 0$ for $k \leq -1$.

We consider a finite-dimensional subspace $W \subseteq \mathfrak{F}_k$ that is *invariant* under T, i.e $f(\tau+1) \in W$ whenever $f(\tau) \in W$. We introduce the $m \times m$ matrix

$$J_{m,\lambda} = \begin{pmatrix} \lambda & & & \\ \lambda & \ddots & & \\ & \ddots & \ddots & \\ & & \lambda & \lambda \end{pmatrix}, \tag{5}$$

i.e. $J_{i,j} = \lambda$ for $i = j$ or $j+1$ and $J_{i,j} = 0$ otherwise.

Lemma 2.1 *There is a basis of W with respect to which the matrix $\rho(T)$ representing T is in block diagonal form*

$$\rho(T) = \begin{pmatrix} J_{m_1,\lambda_1} & & \\ & \ddots & \\ & & J_{m_t,\lambda_t} \end{pmatrix}. \tag{6}$$

Proof: The existence of such a representation is basically the theory of the Jordan canonical form. The *usual* Jordan canonical form is similar to the above, except that the subdiagonal of each block then consists of 1's rather than λ's. The λ's that appear in (6) are the eigenvalues of $\rho(T)$, and in particular they are nonzero on account of the invertibility of $\rho(T)$. Then it is easily checked that (6) is indeed similar to the usual Jordan canonical form, and the Lemma follows. $\square$

We refer to (6) as the *modified Jordan canonical form* of $\rho(T)$, and J_{m_i,λ_i} as a *modified Jordan block*. To a certain extent at least, Lemma 2.1 reduces the study of the functions in W to those associated to one of the Jordan blocks. In this case we have the following basic result.

Theorem 2.2 *Let $W \subseteq \mathfrak{F}_k$ be a T-invariant subspace of dimension m. Suppose that W has an ordered basis $(g_0(\tau), \dots, g_{m-1}(\tau))$ with respect to which the matrix $\rho(T)$ is a single modified Jordan block $J_{m,\lambda}$. Set $\lambda = e^{2\pi i\mu}$. Then there are m convergent q-expansions $h_t(\tau) = \sum_{n\in\mathbb{Z}} a_t(n)q^{n+\mu}, 0 \leq t \leq m-1$, such that*

$$g_j(\tau) = \sum_{t=0}^{j} \binom{\tau}{t} h_{j-t}(\tau), \ 0 \leq j \leq m-1. \tag{7}$$

The case $m = 1$ of the Theorem is well known. We will need it for the proof of the general case, so we state it as

Lemma 2.3 *Let $\lambda = e^{2\pi i\mu}$, and suppose that $f(\tau) \in \mathfrak{F}$ satisfies $f(\tau+1) = \lambda f(\tau)$. Then $f(\tau)$ is represented by a convergent q-expansion*

$$f(\tau) = \sum_{n\in\mathbb{Z}} a(n)q^{n+\mu}. \tag{8}$$

$\square$

Turning to the proof of the Theorem, we have

$$g_j(\tau+1) \quad = \quad \lambda(g_j(\tau) + g_{j-1}(\tau)),\ 0 \le j \le m-1, \tag{9}$$

where $g_{-1}(\tau) = 0$. Set

$$h_j(\tau) = \sum_{t=0}^{j} (-1)^t \binom{\tau+t-1}{t} g_{j-t}(\tau),\ 0 \le j \le m-1.$$

These equalities can be displayed as a system of equations. Indeed,

$$B_m(\tau)\begin{pmatrix} g_0(\tau) \\ \vdots \\ g_{m-1}(\tau) \end{pmatrix} = \begin{pmatrix} h_0(\tau) \\ \vdots \\ h_{m-1}(\tau) \end{pmatrix}, \tag{10}$$

where $B_m(x)$ is the $m \times m$ lower triangular matrix with

$$B_m(x)_{ij} = (-1)^{i-j}\binom{x+i-j-1}{i-j}. \tag{11}$$

Then $B_m(x)$ is invertible and

$$B_m(x)^{-1}_{ij} = \binom{x}{i-j}. \tag{12}$$

We will show that each $h_j(\tau)$ has a convergent q-expansion. This being the case, (7) holds and the Theorem will be proved. Using (9), we have

$$\begin{aligned} h_j(\tau+1) &= \lambda \sum_{t=0}^{j} (-1)^t \binom{\tau+t}{t}(g_{j-t}(\tau) + g_{j-t-1}(\tau)) \\ &= \lambda\left\{ \sum_{t=0}^{j}(-1)^t\left(1+\frac{t}{\tau}\right)\binom{\tau+t-1}{t} g_{j-t}(\tau) + \sum_{t=0}^{j}(-1)^t\binom{\tau+t}{t} g_{j-t-1}(\tau)\right\} \\ &= \lambda\left\{ h_j(\tau) + \sum_{t=0}^{j}(-1)^t\binom{\tau+t-1}{t}\frac{t}{\tau} g_{j-t}(\tau) + \sum_{t=0}^{j}(-1)^t\binom{\tau+t}{t} g_{j-t-1}(\tau)\right\}. \end{aligned}$$

But the sum of the second and third terms in the braces vanishes, being equal to

$$\begin{aligned} &\sum_{t=1}^{j}(-1)^t\binom{\tau+t-1}{t}\frac{t}{\tau}g_{j-t}(\tau) + \sum_{t=1}^{j}(-1)^{t-1}\binom{\tau+t-1}{t-1}g_{j-t}(\tau) \\ = \quad &\sum_{t=1}^{j}(-1)^{t-1}g_{j-t}(\tau)\left\{\binom{\tau+t-1}{t-1} - \binom{\tau+t-1}{t}\frac{t}{\tau}\right\} = 0. \end{aligned}$$

Thus we have established the identity $h_j(\tau+1) = \lambda h_j(\tau)$. By Lemma 2.3, $h_j(\tau)$ is indeed represented by a q-expansion of the desired shape, and the proof of Theorem 2.2 is complete. $\square$

We call (7) a *polynomial* q-expansion. The space of polynomials spanned by $\binom{x}{t}, 0 \le t \le m-1$ is also spanned by the powers $x^t, 0 \le t \le m-1$. Bearing in mind that $(2\pi i\tau)^t = (\log q)^t$, it follows that in Theorem 2.2 we can find a basis $\{g'_j(\tau)\}$ of W such that

$$g'_j(\tau) = \sum_{t=0}^{j} (\log q)^t h'_{j-t}(\tau) \tag{13}$$

with $h'_t(\tau) = \sum_{n\in\mathbb{Z}} a'_t(n)q^{n+\mu}$. We refer to (13) as a *logarithmic* q-expansion.

2.3 Logarithmic vector-valued modular forms

We say that a function $f(\tau)$ with a q-expansion (8) is *meromorphic at infinity* if

$$f(\tau) = \sum_{n+\Re(\mu)\ge n_0} a(n)q^{n+\mu}.$$

That is, the Fourier coefficients $a(n)$ *vanish* for exponents $n+\mu$ whose *real parts* are small enough. A polynomial (or logarithmic) q-expansion (7) is holomorphic at infinity if each of the associated ordinary q-expansions $h_{j-t}(\tau)$ are holomorphic at infinity. Similarly, $f(\tau)$ *vanishes* at ∞ if the Fourier coefficients $a(n)$ vanish for $n+\Re(\mu) \le 0$; a polynomial q-expansion vanishes at ∞ if the associated ordinary q-expansions vanish at ∞. These conditions are independent of the chosen representations.

Now assume that $F(\tau) = (f_1(\tau), \dots, f_p(\tau))^t$ is an unrestricted vector-valued modular form of weight k with respect to ρ. It follows from (4) that the span W of the functions $f_j(\tau)$ is a right Γ-submodule of $\mathfrak{F}$ satisfying $f_j(\tau+1) \in W$. Choose a basis of W so that $\rho(T)$ is in modified Jordan canonical form. By Theorem 2.2 the basis of W consists of functions $g_j(\tau)$ which have polynomial q-expansions. We call $F(\tau)$, or (F,ρ), a *logarithmic meromorphic, holomorphic, or cuspidal vector-valued modular form* respectively if each of the functions $g_j(\tau)$ is meromorphic, is holomorphic, or vanishes at ∞, respectively.

We let $\mathcal{H}(k,\rho)$ denote the holomorphic LVVMFs of weight k with respect to ρ. It is finite-dimensional complex vector space ([KM3]).

3 Polynomial-growth estimate of the Fourier coefficients

3.1 The new estimate

We state a modification and elaboration of ([E], p. 169, displays (3)-(5)) which we call *Eichler's canonical representation* for elements of Γ:

Lemma 3.1 *Let $\gamma \in \Gamma, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$. We may assume without loss of generality that $c > 0$. Then*

(a) *γ has a unique representation*

$$\gamma = (ST^{l_{\nu+1}}) \dots (ST^{l_1})(ST^{l_0}) \tag{14}$$

where $(-1)^{j-1} l_j > 0$ for $1 \leq j \leq \nu$ and $(-1)^\nu l_{\nu+1} \geq 0$. Thus l_1 is positive, the l_j alternate in sign for $j \geq 1$ (with the proviso that $l_{\nu+1}$ may be zero) and there is no condition on l_0.

(b) *$l_{\nu+1} \neq 0$ if, and only if, $|a/c| < 1$; in the opposite case $|a/c| \geq 1$ (whence $l_{\nu+1} = 0$), we have $l_\nu = \pm[|a/c|]$.*

□

Remark 3.2 *1. Eichler does not state (14) precisely as we have here, but his result is the same. The proof, omitted in [E], entails repeated application of the division algorithm in $\mathbb{Z}$.*
2. [E] makes no mention of part (b). However, that it holds is clear from the proof mentioned in Remark 1.

Now, let $\gamma \in \Gamma$ be fixed as in Lemma 3.1, with canonical representation (14). We set

$$\begin{aligned} P_0 &= ST^{l_0}, \\ P_{j+1} &= (ST^{l_{j+1}})P_j, \ 0 \leq j \leq \nu, \\ P_j &= \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}, \ 0 \leq j \leq \nu + 1. \end{aligned} \tag{15}$$

Proposition 3.3 *(a) Assume $l_{\nu+1} \neq 0$. Then we have*

$$\begin{aligned} |l_0 l_1 \dots l_{\nu+1}| &\leq |d| && \text{if } l_0 < 0; \\ |l_1 \dots l_{\nu+1}| &\leq |d - c| && \text{if } l_0 = 0; \\ |l_0 l_1 \dots l_{\nu+1}| &\leq |c| + |d| && \text{if } l_0 > 0. \end{aligned} \tag{16}$$

(b) If $l_{\nu+1} = 0$, then

$$\begin{aligned} |l_0 l_1 \dots l_{\nu-1}| &\leq |d| && \text{if } l_0 < 0; \\ |l_1 \dots l_{\nu-1}| &\leq |d - c| && \text{if } l_0 = 0; \\ |l_0 l_1 \dots l_{\nu-1}| &\leq |c| + |d| && \text{if } l_0 > 0. \end{aligned} \tag{17}$$

Proof: (a). Assume $l_0 < 0$. We will prove by induction on $j \geq 0$ that

$$\begin{aligned} &(i)\ |l_0 l_1 \dots l_j| \leq |d_j|, \\ &(ii)\ (-1)^j b_j d_j \geq 0. \end{aligned} \tag{18}$$

Once this is established, the case $j = \nu + 1$ of (18)(i) proves (16) in this case. Now

$$P_0 = \begin{pmatrix} 0 & -1 \\ 1 & l_0 \end{pmatrix},$$

and the case $j = 0$ is clear. For the inductive step, we have

$$P_{j+1} = \begin{pmatrix} 0 & -1 \\ 1 & l_{j+1} \end{pmatrix} \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} = \begin{pmatrix} -c_j & -d_j \\ a_j + l_{j+1}c_j & b_j + l_{j+1}d_j \end{pmatrix}. \tag{19}$$

Thus $(-1)^{j+1}b_{j+1}d_{j+1} = (-1)^j b_j d_j + (-1)^j l_{j+1} d_j^2 \geq 0$ where the last inequality uses induction and the inequality stated in Lemma 3.1. So (18)(ii) holds.

As for (18)(i), note that because $(-1)^j b_j d_j$ and $(-1)^j l_{j+1} d_j^2$ are both nonnegative then b_j and $l_{j+1}d_j$ have the *same sign*. Therefore using induction again, we have $|l_0 l_1 \ldots l_{j+1}| \leq |d_j l_{j+1}| \leq |b_j| + |l_{j+1}d_j| = |b_j + l_{j+1}d_j| = |d_{j+1}|$. This completes the proof in the case $l_0 < 0$.

Assume $l_0 = 0$. Notice that

$$\gamma T^{-1} = (ST^{l_{\nu+1}}) \ldots (ST^{l_1})(ST^{-1})$$

is an instance of the first case, with $l_0 = -1$. Since

$$\gamma T^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b-a \\ c & d-c \end{pmatrix},$$

it follows from the case $l_0 < 0$ that $|l_1 \ldots l_{\nu+1}| \leq |d - c|$, as was to be proved.

Now suppose $l_0 > 0$. We will prove by induction on j that

$$\begin{aligned} &(i)\ |l_0 l_1 \ldots l_j| \leq |c_j| + |d_j|,\ j \geq 0 \\ &(ii)\ (-1)^j b_j d_j, (-1)^j a_j c_j \geq 0,\ j \geq 1. \end{aligned} \tag{20}$$

Once again, the case $j = \nu + 1$ of (20)(i) proves the third case of (16). Now

$$P_0 = \begin{pmatrix} 0 & -1 \\ 1 & l_0 \end{pmatrix}, P_1 = \begin{pmatrix} -1 & -l_0 \\ l_1 & l_0 l_1 - 1 \end{pmatrix}.$$

So when $j = 0$, (20)(i) is clearly true, and because $l_0, l_1 > 0$ we also have

$$-a_1 c_1 = l_1 > 0,\ -b_1 d_1 = l_0(l_0 l_1 - 1) \geq 0.$$

So (20)(ii) holds for $j = 1$. As for the inductive step, P_{j+1} is given by (19), and the proof that $(-1)^j b_j d_j \geq 0$ is the same as in the case $l_0 < 0$. Similarly, $(-1)^{j+1} a_{j+1} c_{j+1} = (-1)^j c_j a_j + (-1)^j l_{j+1} c_j^2 \geq 0$ is the sum of two nonnegative terms and hence is itself nonnegative, so (20)(ii) holds. Finally, by an argument similar to that used when $l_0 < 0$, we have $|l_0 \ldots l_{j+1}| \leq |c_j + d_j||l_{j+1}| < |l_{j+1}c_j| + |l_{j+1}d_j| + |a_j| + |b_j| = |a_j + l_{j+1}c_j| + |b_j + l_{j+1}d_j| = |c_{j+1}| + |d_{j+1}|$. Part (a) of the Proposition is proved.

(b). When $l_{\nu+1} = 0$, note that

$$\gamma = -T^{l_\nu}(ST^{l_{\nu-1}}) \ldots (ST^{l_1})(ST^{l_0}) \tag{21}$$

and $l_\nu \neq 0$, so that the argument of part (a) applies to $-T^{-l_\nu}\gamma$ rather than to γ itself. Since

$$-T^{-l_\nu}\gamma = \begin{pmatrix} * & * \\ -c & -d \end{pmatrix},$$

we obtain the inequalities (17). This completes the proof of the Proposition. □

The *Eichler length* of γ with canonical representation (14) is given by

$$L(\gamma) = \begin{cases} 2\nu+4, & l_0 l_{\nu+1} \neq 0, \\ 2\nu+3, & l_0 = 0, l_{\nu+1} \neq 0, \\ 2\nu+1, & l_0 \neq 0, l_{\nu+1} = 0, \\ 2\nu, & l_0 = l_{\nu+1} = 0. \end{cases} \tag{22}$$

(See (21) above.)

By Lamé's Theorem we have the estimate

$$L(\gamma) \leq K(\log|c| + 1) \tag{23}$$

with a positive constant K independent of γ. (Cf. [E], p.179.)

3.2 The matrix norm

The *norm* $||\rho(\gamma)||$, defined to be

$$\max_{i,j} |\rho(\gamma)_{ij}|$$

satisfies the multiplicative condition

$$||\rho(\gamma_1\gamma_2)|| \leq p||\rho(\gamma_1)||||\rho(\gamma_2)|| \quad (\gamma_1, \gamma_2 \in \Gamma), \tag{24}$$

where $p = \dim \rho$. Let $\gamma \in \Gamma$ be expressed in the canonical form (14). Again there are two cases to consider, according as $l_{\nu+1} \neq 0$ or $l_{\nu+1} = 0$. If $l_{\nu+1} \neq 0$, then by (24),

$$||\rho(\gamma)|| \leq p^{2\nu+2}||\rho(S)||^{\nu+2} \prod_{j=0}^{\nu+1} ||\rho(T^{l_j}||. \tag{25}$$

If $l_{\nu+1} = 0$, then (14) reduces to (21), so that (24) implies

$$||\rho(\gamma)|| \leq p^{2\nu+1}||\rho(S)||^{\nu} \prod_{j=0}^{\nu} ||\rho(T^{l_j})||.$$

Since $\rho(T^{l_{\nu+1}}) = \rho(I) = I$ in this case, we obtain the upper estimate

$$||\rho(\gamma)|| \leq Kp^{2\nu+1}||\rho(S)||^{\nu+1} \prod_{j=0}^{\nu+1} ||\rho(T^{l_j})|| \tag{26}$$

in both cases. In (26), K is a constant depending only on ρ.

Lemma 3.4 *Let s be the maximum of the sizes m_j of the Jordan blocks J_{m_j,λ_j} of $\rho(T)$ (5), (6). There is a constant C_s depending only on s such that for $l \neq 0$,*

$$||\rho(T^l)|| \leq C_s|l|^{s-1}. \tag{27}$$

Proof: We have

$$J_{m,\lambda}^l = \lambda^l J_{m,1}^l = \lambda^l (I_m + N)^l = \lambda^l \sum_{i \geq 0} \binom{l}{i} N^i$$

where N is the nilpotent $m \times m$ matrix with each $(i, i-1)$-entry equal to 1 ($i \geq 2$), and all other entries zero. Note that $N^m = 0$ and the entries of N^i for $1 \leq i < m$ are 1 on the ith. subdiagonal and zero elsewhere. Bearing in mind that $|\lambda| = 1$, we see that $||J_{m,\lambda}^l||$ is majorized by the maximum of the binomial coefficients $\binom{l}{i}$ over the range $0 \leq i \leq m-1$. Since $\binom{l}{i}$ is a polynomial in l of degree i then we certainly have $||J_{m,\lambda}^l|| \leq C_m |l|^{m-1}$ for a universal constant C_m, and since this applies to each Jordan block of $\rho(T^l)$ then the Lemma follows immediately. $\square$

Corollary 3.5 *There are universal constants K_3, K_4 such that*

$$||\rho(\gamma)|| \leq \begin{cases} K_3(c^2+d^2)^{K_4}, & l_{\nu+1} \neq 0, \\ K_3(c^2+d^2)^{K_4} |l_\nu|^{s-1}, & l_{\nu+1} = 0. \end{cases} \tag{28}$$

Moreover the same *estimates hold for $||\rho(\gamma^{-1})||$.*

Proof: First assume that $l_{\nu+1} \neq 0$. From Lemma 3.4 and (26) we obtain

$$||\rho(\gamma)|| \leq \begin{cases} K_1^{\nu+1} \prod_{j=0}^{\nu+1} |l_j|^{s-1}, & l_0 \neq 0, \\ K_1^{\nu+1} \prod_{j=1}^{\nu+1} |l_j|^{s-1}, & l_0 = 0, \end{cases}$$

for a constant K_1 depending only on ρ. Now use (22), (23) and Proposition 3.3(a) to see that

$$||\rho(\gamma)|| \leq e^{(\log K_1) K_2 \log(|c|+1)} (|c| + |d|)^{s-1} \leq K_3(c^2+d^2)^{K_4}.$$

Concerning the second assertion of the Corollary, since

$$\gamma^{-1} = (T^{-l_0} S)(T^{-l_1} S) \ldots (T^{-l_{\nu+1}} S)$$

we have (26) again, but with T^{l_j} replaced by T^{-l_j}. The rest of the proof is identical to the proof of the estimate of $||\rho(\gamma)||$, so that we indeed obtain estimate (28) for γ^{-1} as well as γ.

$$||\rho(\gamma^{-1})|| \leq ||\rho(S)||^{\nu+2} \prod_{j=0}^{\nu+1} ||\rho(T)^{-l_j}||,$$

and (27) then holds by Lemma 3.4. The rest of the proof is identical to the previous case, so that we indeed obtain the estimate (28) for γ^{-1} as well as γ.

The second case, in which $l_{\nu+1} = 0$, is analogous. In this case we apply Proposition 3.3(b) in place of Proposition 3.3(a). $\square$

3.3 Application to the Fourier coefficients

Let $F(\tau) \in \mathcal{H}(k,\rho)$ be a logarithmic vector-valued modular form of weight k. We are going to show that the Fourier coefficients of $F(\tau)$ satisfy a polynomial growth condition for $n \to \infty$. Let $F(\tau) = (f_1(\tau), \ldots, f_p(\tau))^t$ with

$$f_l(\tau) = \sum_{u=0}^{l} \binom{\tau}{u} h_{l-u}(\tau), \quad 0 \le l \le m_j - 1.$$

Here, we have relabelled the components in the jth. block for notational convenience.

The proof is similar to the case treated in [KM1], but with an additional complication due to the fact that we are dealing with polynomial q-expansions rather than ordinary q-expansions. To deal with this we make use of the estimates that we have obtained in Subsection 3.1. We continue to assume that the eigenvalues of $\rho(T)$ are of absolute value 1. We will sometimes drop the subscript j from the notation when it is convenient.

We write $\tau = x + iy$ for $\tau \in \mathfrak{H}$ and let $\mathfrak{R}$ be the usual fundamental region for Γ. Write $z = u + iv$ for $z \in \overline{\mathfrak{R}}$. Choose a real number $\sigma > 0$ to be fixed later, and set

$$g_l(\tau) = y^\sigma |f_l(\tau)|.$$

Because $F(\tau)$ is holomorphic, $a_l(n) = 0$ unless $n + \mu \ge 0$. It follows that there is a constant K_1 such that

$$g_l(z) \le K_1 v^{\delta\sigma}, \quad 1 \le l \le p, \ z \in \overline{\mathfrak{R}}, \tag{29}$$

where $\delta = 0$ if $F(\tau)$ is a *cusp-form*, and is 1 otherwise.

Choose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, set $\tau = \gamma z$ ($z \in \overline{\mathfrak{R}}$), and write γ in Eichler canonical form (14). We wish to argue just as in [KM1], pp.121-122, and to do this we need to make use of Proposition 3.3, a feature of the proof not required in the normal case (loc. cit.).

We have, for $\tau \in \mathfrak{H}$ and γ, z as above,

$$\begin{aligned} g_l(\tau) &= g_l(\gamma z) = (v|cz+d|^{-2})^\sigma |f_l(\gamma z)| \\ &= v^\sigma |cz+d|^{k-2\sigma} |(f_l|_k \gamma)(z)| \\ &= l^{th} \text{ component of } v^\sigma |cz+d|^{k-2\sigma} |\rho(\gamma) F(z)| \\ &= v^\sigma |cz+d|^{k-2\sigma} \left| \sum_{m=1}^{p} \rho(\gamma)_{lm} f_m(z) \right| \\ &= |cz+d|^{k-2\sigma} \left| \sum_{m=1}^{p} \rho(\gamma)_{lm} g_m(z) \right|. \end{aligned}$$

This then implies (by the triangle inequality)

$$g_l(\tau) \le |cz+d|^{k-2\sigma} \sum_{m=1}^{p} |\rho(\gamma)_{lm}| g_m(z).$$

Since $z \in \overline{\mathfrak{R}}$, we also know ([KM1], display (13)) that

$$c^2 + d^2 \leq K_6 |cz + d|^2 \tag{30}$$

for a universal constant K_6. Using (29), (30) and Corollary 3.5, we obtain

$$\begin{aligned} g_l(\tau) &\leq K_1 v^{\delta\sigma} |cz+d|^{k-2\sigma} \sum_{m=1}^{p} |\rho(\gamma)_{lm}| \\ &\leq \begin{cases} K_2 v^{\delta\sigma}|cz+d|^{k-2\sigma}(c^2+d^2)^{K_4}, & l_{\nu+1} \neq 0, \\ K_2 v^{\delta\sigma}|cz+d|^{k-2\sigma}(c^2+d^2)^{K_4}|l_\nu|^{s-1}, & l_{\nu+1} = 0, \end{cases} \\ &\leq \begin{cases} K_2' v^{\delta\sigma}|cz+d|^{k-2\sigma+K_5}, & l_{\nu+1} \neq 0, \\ K_2' v^{\delta\sigma}|cz+d|^{k-2\sigma+K_5}|l_\nu|^{s-1}, & l_{\nu+1} = 0. \end{cases} \end{aligned}$$

Choosing $\sigma = (k + K_5)/2$ leads to

$$g_l(\tau) \leq \begin{cases} K_2' v^{\delta(k+K_5)/2}, & l_{\nu+1} \neq 0, \\ K_2' v^{\delta(k+K_5)/2}|l_\nu|^{s-1}, & l_{\nu+1} = 0. \end{cases}$$

In the cuspidal case we have $\delta = 0$, whence $g_l(\tau)$ is *bounded* in $\mathfrak{H}$, by a universal constant K_6 if $l_{\nu+1} \neq 0$, and by $K_6 |l_\nu|^{s-1}$ if $l_{\nu+1} = 0$. Then

$$|f_l(\tau)| = y^{-\sigma} g_l(\tau) = \begin{cases} O(y^{-(k+K_5)/2}), & l_{\nu+1} \neq 0, \\ O(y^{-(k+K_5)/2})|l_\nu|^{s-1}, & l_{\nu+1} = 0. \end{cases}$$

In the first case, when $l_{\nu+1} \neq 0$, a standard argument, entailing integration on the interval $\tau = x + i/n$ ($n \in \mathbb{Z}^+, |x| \leq 1/2$) implies that the Fourier coefficients of $f_l(\tau)$ satisfy $a(n) = O(n^{(k+K_5)/2})$ for $n \to \infty$. In the second case, when $l_{\nu+1} = 0$, an elementary argument using the location of z and τ ($z \in \overline{\mathfrak{R}}, \tau = x + i/n, n \in \mathbb{Z}^+$) implies that $|a/c| < 2$. By Lemma 3.1)(b), then, if we keep in mind that $l_\nu \neq 0$ it follows that $|l_\nu| = [|a/c|] = 1$. Hence the argument used in the case $l_{\nu+1} \neq 0$ implies again in this case that $a(n) = O(n^{(k+K_5)/2})$ for $n \to \infty$. □

In the holomorphic (noncuspidal) case there is a similar argument (cf. [KM1], p. 123) wherein the exponent is doubled. We have proved

Theorem 3.6 *Let ρ be a representation of Γ such that all eigenvalues of $\rho(T)$ lie on the unit circle, and suppose that $F(\tau) \in \mathcal{H}(k, \rho)$. There is a constant α depending only on ρ such that the Fourier coefficients of $F(\tau)$ satisfy $a(n) = O(n^{k+\alpha})$ for $n \to \infty$. If $F(\tau)$ is cuspidal then $a(n) = O(n^{(k+\alpha)/2})$ for $n \to \infty$.* □

Errata. We take this opportunity to correct a few typographical errors in [KM3], upon which the present paper is based.
p.271, ll -14/-13. This should read as follows. 'Here $\mathcal{M}^*$ is the set of cosets of $\Gamma_\infty \backslash \Gamma$ distinct from $\pm\langle T \rangle$, where Γ_∞ is the stabilizer of ∞ in Γ, and ...'
p.272, l-3. This should be $P_{j+1} = (ST^{l_{j+1}})P_j, \ 0 \leq j \leq \nu$,
p.274, l-5. The right-hand side of display (36) should be $p^{2\nu+2}||\rho(S)||^{\nu+2} \prod_{j=0}^{\nu+1} ||\rho(T^{l_j})||$.
p.274, l10. Replace j by $j+1$.

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